

→ This is nothing but the rotation around \hat{z} -axis
with angle $\phi = \omega t$!

But, there's a weird thing.

• $\langle \vec{S} \rangle_{2\pi} = \langle \vec{S} \rangle_0$; [t's ok.]

$$|\alpha, t\rangle = U(t) [|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|] |\alpha\rangle \quad || \phi = \omega t.$$

$$= e^{-\frac{i\phi}{2}} |\uparrow\rangle\langle\uparrow|\alpha\rangle + e^{\frac{i\phi}{2}} |\downarrow\rangle\langle\downarrow|\alpha\rangle$$

★ ★ ★ $|\alpha, 2\pi\rangle = \underbrace{-}_{\text{wavy}} |\alpha, 0\rangle.$!!!

The state comes back with a minus sign!

or.
↳ precession period $T = \frac{2\pi}{\omega}$ for $\langle \vec{S} \rangle$.

but $T_{\text{stateket}} = \frac{4\pi}{\omega}$ for $|\alpha\rangle$.

(3) Generalization: SU(2) vs. SO(3)

• Pauli two-component formalism
with the "Pauli" spinor.

ket
bra

$$|\uparrow\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \chi_{\uparrow}, \quad |\downarrow\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \chi_{\downarrow}$$

$$\langle\downarrow| \doteq (1, 0) \equiv \chi_{\uparrow}^{\dagger}, \quad \langle\uparrow| \doteq (0, 1) \equiv \chi_{\downarrow}^{\dagger}$$

a state

$$|\alpha\rangle \doteq \begin{pmatrix} \langle\uparrow|\alpha\rangle \\ \langle\downarrow|\alpha\rangle \end{pmatrix}, \quad \langle\alpha| \doteq (\langle\alpha|\uparrow\rangle, \langle\alpha|\downarrow\rangle).$$

\Rightarrow two-component "Pauli" spinor.

$$\underline{\chi} = \begin{pmatrix} \langle \uparrow | \alpha \rangle \\ \langle \downarrow | \alpha \rangle \end{pmatrix} \equiv \begin{pmatrix} c_{\uparrow} \\ c_{\downarrow} \end{pmatrix} = c_{\uparrow} \chi_{\uparrow} + c_{\downarrow} \chi_{\downarrow}$$

and

$$\underline{\chi}^{\dagger} = (\langle \alpha | \uparrow \rangle, \langle \alpha | \downarrow \rangle) = (c_{\uparrow}^*, c_{\downarrow}^*)$$

- Pauli Matrices

def. $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\Rightarrow \underline{\tilde{S}}_k \equiv \frac{\hbar}{2} \sigma_k$$

ex. $\langle S_k \rangle = \langle \alpha | S_k | \alpha \rangle = \frac{\hbar}{2} \chi^{\dagger} \sigma_k \chi$ ← Try to verify this.

\Rightarrow properties :

$$\begin{cases} \sigma_i^2 = 1 \\ \{\sigma_i, \sigma_j\} = 2\delta_{ij} \\ [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \end{cases}$$

also,

$$\begin{cases} \sigma_i^{\dagger} = \sigma_i & : \text{Hermitian.} \\ \det[\sigma_i] = -1 & : \text{"special"} \\ \text{Tr}[\sigma_i] = 0 & : \text{traceless.} \end{cases}$$

Now, consider a vector $\vec{X} = (x, y, z)$ in the basis of $\in \mathbb{R}$

Pauli matrices :

$$\underline{X} = x\sigma_1 + y\sigma_2 + z\sigma_3$$

$$= \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}$$

: Hermitian, traceless.

length of the vector $|\vec{x}|^2 = x^2 + y^2 + z^2 = \underline{-\det X}$. 19

→ A rotation can be described by a unitary transformation,

$$X' = U X U^{-1}, \quad \parallel \underline{\det U = 1}$$

$$X' = \vec{x}', \vec{\sigma}$$

$$\Rightarrow |\vec{x}'|^2 = |\vec{x}|^2 \Leftrightarrow \underline{\det X' = \det X} //$$

∴ U (a 2×2 matrix) is a rotation matrix,

mapping $SU(2) [U]$ onto $SO(3) [R]$.

special \leftarrow $\begin{matrix} \text{unitary} \\ \text{dimension of} \end{matrix}$ "defining" representation, $\rightarrow 2 \times 2$ matrix or "fundamental"

$$\therefore \det U = 1$$

Since U is a 2×2 matrix, it can be written as

$$U = q_0 + i \vec{\sigma} \cdot \vec{q} \quad \parallel \vec{q} = (q_1, q_2, q_3).$$

see HW 2.1

$$U U^\dagger = 1 \Rightarrow |q_0|^2 + |\vec{q}|^2 + i \vec{\sigma} \cdot (\vec{q} q_0 - \text{c.c.}) + i \vec{\sigma} \cdot (\vec{q} \times \vec{q}^*) = 1$$

* Use the identity $(\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = \vec{a} \cdot \vec{b} + i \vec{\sigma} \cdot (\vec{a} \times \vec{b})$.

$\Rightarrow q_0$ and \vec{q} are real. (to remove $\vec{\sigma}$ -dependence)
chosen to be.

$$q_0^2 + |\vec{q}|^2 = 1 //$$

$$\text{Choosing } q_0 = \cos \frac{\theta}{2}, \quad \vec{q} = -\hat{z} \sin \frac{\theta}{2},$$

$$U X U^{-1} \rightarrow \begin{pmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \\ z \end{pmatrix}$$

[It rotates \vec{x} by θ around z -axis.]

check: $U = \cos \frac{\theta}{2} - i \hat{n} \cdot \vec{\sigma} \sin \frac{\theta}{2} = \exp \left[-i \frac{\vec{\sigma} \cdot \hat{n}}{2} \theta \right] = \exp \left[-i \frac{\vec{J}}{\hbar} \theta \right]$

a general rotation by angle ϕ around \hat{n} -axis.

$$: \quad q_0 = \cos \frac{1}{2} \phi, \quad \vec{q} = -\hat{n} \sin \frac{1}{2} \phi$$

$$\rightarrow U = \exp \left[-i \frac{1}{2} (\vec{\sigma} \cdot \hat{n}) \phi \right] \quad \parallel \quad \text{This is the case where } \vec{J} = \frac{\hbar}{2} \vec{\sigma}$$

Verification

$$U = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \left(\frac{\phi}{2} \right)^n (\hat{n} \cdot \vec{\sigma})^n \quad \text{by using } (\vec{\sigma} \cdot \vec{a})(\vec{\sigma} \cdot \vec{b}) = \vec{a} \cdot \vec{b} + i \vec{\sigma} \cdot (\vec{a} \times \vec{b})$$

$$\Rightarrow (\hat{n} \cdot \vec{\sigma})^{2n} = 1$$

$$= \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{\phi}{2} \right)^{2k} \right] \cdot I - i (\hat{n} \cdot \vec{\sigma}) \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{\phi}{2} \right)^{2k+1} \right]$$

$$= \cos \frac{\phi}{2} \cdot I - i (\hat{n} \cdot \vec{\sigma}) \sin \frac{\phi}{2}$$

* U has the period of 4π ! Does it sound reasonable?

Yes. $SU(2)$ covers $SO(3)$ twice!

"Cayley-Klein" parameters

$$\text{In another general form, } U(a, b) = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

$$\text{with } |a|^2 + |b|^2 = 1$$

$$\Rightarrow U^\dagger(a, b) X U(a, b) = X'$$

$$\parallel \begin{aligned} U(-a, b) \\ = -U(a, b) \end{aligned}$$

$$U^\dagger(-a, -b) X U(-a, -b) = X'$$

U and $-U$ generates the same R .

$$[2\pi] + [2\pi] \longrightarrow [2\pi]$$

\therefore A state let rotated by U has 4π -periodicity!

$$= U(\hat{n}, \phi) |\alpha\rangle$$

in $SU(2)$.

(4) Eigenvalues and Eigenstates of \vec{J} and J^2 .

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• Commutation Relations and Ladder Operators

Lie Algebra: $[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$

Every thing starts from this relation.

Casimir operator

$$\Rightarrow [J^2, J_k] = 0 \quad || \quad J^2 = J_x^2 + J_y^2 + J_z^2$$

: There are simultaneous eigenkets of J^2 and J_k .

$$\Rightarrow J^2 |a, b\rangle = a |a, b\rangle$$

$$J_z |a, b\rangle = b |a, b\rangle$$

NOTE: There's a typo in SQN, 2nd ed.

def. Ladder operators : let's see how these work.

$$J_{\pm} \equiv J_x \pm i J_y$$

Commutation relations: $[J_+, J_-] = 2\hbar J_z$

$$[J_z, J_{\pm}] = \pm \hbar J_{\pm}$$

Why "ladder"?

$$[J^2, J_{\pm}] = 0.$$

$$\begin{aligned} \Rightarrow J_z (J_{\pm} |a, b\rangle) &= ([J_z, J_{\pm}] + J_{\pm} J_z) |a, b\rangle \\ &= (b \pm \hbar) (J_{\pm} |a, b\rangle) \end{aligned}$$

: It raises or lowers the eigenvalue b .

But it doesn't change "a" since $[J^2, J_{\pm}] = 0$.

$$\Rightarrow J^2 (J_{\pm} |a, b\rangle) = a (J_{\pm} |a, b\rangle)$$

Therefore, we may write it as

$$J_{\pm} |a, b\rangle = c_{\pm} |a, b \pm \hbar\rangle$$

c-number.

• Eigenvalues of J^2 and J_z .

Can we apply J_{\pm} again and again, indefinitely? NO.

$$\begin{aligned} \text{Consider } J^2 - J_z^2 &= \frac{1}{2} (J_+ J_- + J_- J_+) \\ &= \frac{1}{2} (J_-^{\dagger} J_- + J_+^{\dagger} J_+) \end{aligned}$$

$$\Rightarrow \langle a, b | J^2 - J_z^2 | a, b \rangle = \frac{1}{2} [\langle - | - \rangle + \langle + | + \rangle]$$

$$\geq 0 \quad \parallel \quad \begin{aligned} | - \rangle &= J_- | a, b \rangle \\ | + \rangle &= J_+ | a, b \rangle \end{aligned}$$

$$\Rightarrow \underline{a \geq b^2} \quad \parallel \quad b \text{ has upper and lower bounds given by } a.$$

$$\text{Thus, } J_+ | a, b_{\max} \rangle = 0 \Rightarrow J_- J_+ | a, b_{\max} \rangle = 0.$$

$$\Rightarrow \underline{(J^2 - J_z^2 - \hbar J_z) | a, b_{\max} \rangle = 0}$$

$$\text{proof: } J_- J_+ = J_x^2 + J_y^2 - \hbar J_z = J^2 - J_z^2 - \hbar J_z$$

$$\Rightarrow a - b_{\max}^2 - b_{\max} \hbar = 0$$

$$\text{or } a = b_{\max} (b_{\max} + \hbar) \quad \dots \textcircled{1}$$

$$\text{Similarly, } J_- | a, b_{\min} \rangle = 0 \Rightarrow J_+ J_- | a, b_{\min} \rangle = 0.$$

$$\Rightarrow (J^2 - J_z^2 + \hbar J_z) | a, b_{\min} \rangle = 0$$

$$\Rightarrow a = b_{\min} (b_{\min} - \hbar) \quad \dots \textcircled{2}$$

$$\Rightarrow \textcircled{1} - \textcircled{2} : (b_{\max}^2 - b_{\min}^2) + \hbar (b_{\max} + b_{\min}) = 0.$$

$$\Rightarrow \boxed{b_{\min} = -b_{\max}}$$

Since we can reach b_{\max} by applying J_+ to $|b_{\min}\rangle$

a finite number of times, $b_{\max} = b_{\min} + n\hbar$ //

Define $j = \frac{n}{2} = \frac{1}{2}, 1, \frac{3}{2}, \dots$ $n: \text{integer}, > 0$

\Downarrow

$b_{\max} = \frac{n\hbar}{2}$
 $b_{\min} = -\frac{n\hbar}{2}$

\downarrow

let,

$a = \hbar^2 j(j+1)$ and $b \equiv m\hbar$ //

The allowed $m = -j, -j+1, \dots, j-1, j$ //

\therefore $\left[\begin{array}{l} J^2 |j, m\rangle = j(j+1)\hbar^2 |j, m\rangle \\ J_z |j, m\rangle = m\hbar |j, m\rangle \end{array} \right.$ // $j = \frac{1}{2}, 1, \frac{3}{2}, \dots$

\uparrow
a half integer!

\therefore This is a direct outcome of the Lie Algebra;

We did not use anything else.

(b) Matrix elements of \vec{J} and $D(R)$.

• J^2, J_z, J_{\pm}

obviously, $\langle j', m' | J^2 | j, m \rangle = j(j+1)\hbar^2 \delta_{j'j} \delta_{m'm}$

$\langle j', m' | J_z | j, m \rangle = m\hbar \delta_{j'j} \delta_{m'm}$

For J_+ , we know $J_+ |j, m\rangle = C_{jm}^{(+)} |j, m+1\rangle$.

$\Rightarrow \langle j, m | J_+^\dagger J_+ | j, m \rangle = \langle j, m | (J^2 - J_z^2 - \hbar J_z) | j, m \rangle$

$= \hbar^2 [j(j+1) - m^2 - m]$

$\therefore |C_{jm}^{(+)}|^2 = \hbar^2 [j(j+1) - m^2 - m] = \hbar^2 (j-m)(j+m+1)$ //

Choosing $C_{jm}^{(\pm)}$ to be real and positive,

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$$J_{\pm} |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)} \hbar |j, m \pm 1\rangle$$

(You can check this for J_- similarly.)

$$\Rightarrow \langle j', m' | J_{\pm} | j, m \rangle = \sqrt{(j \mp m)(j \pm m + 1)} \hbar \delta_{j', j} \delta_{m', m \pm 1}$$

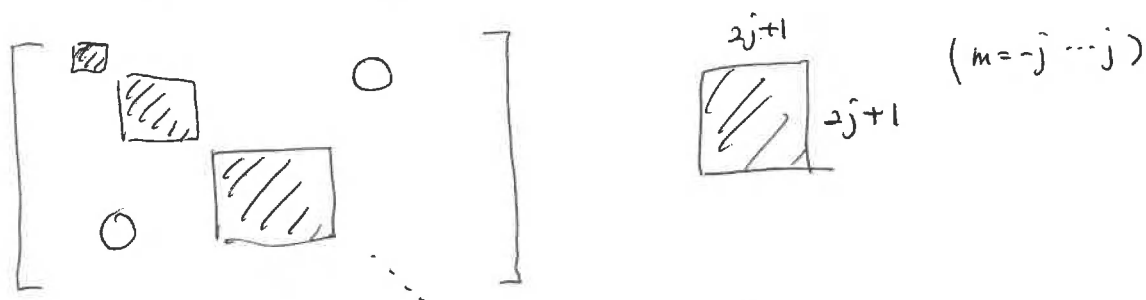
• Representations of the Rotation Operator


$$D_{mm'}^{(j)}(R) \equiv \langle j, m' | \exp\left[-\frac{i}{\hbar} (\vec{J} \cdot \hat{n}) \phi\right] | j, m \rangle$$

(Wigner function) or d-matrix: a matrix element of $D(R)$

NOTE: it's diagonal in $|j\rangle$. $\parallel \vec{J}|j\rangle \propto |j\rangle$

(\Rightarrow a block-diagonal matrix



\rightarrow - $(2j+1)$ by $(2j+1)$ -
The rotation matrices characterized by definite j : 
form a "group".

NOTE: 2 is the dimension of the
on $SU(2)$ "defining, fundamental" rep.

- Identity: $\phi = 0$.

- Inverse: $\phi \rightarrow -\phi$

- Composition:

$$\sum_{m'} D_{m''m'}^{(j)}(R_1) D_{m'm}^{(j)}(R_2) = D_{m''m}^{(j)}(R_1 R_2)$$